

11.1 Let g be a Lorentzian metric on \mathbb{R}^{3+1} with the property that, in the standard Cartesian coordinates (t, x^1, x^2, x^3) on \mathbb{R}^{3+1} , we have

$$|g_{\alpha\beta} - \eta_{\alpha\beta}| < \frac{1}{10}$$

and that the components of the deformation tensor $\pi_{\alpha\beta}^{(T)} = \nabla_\alpha T_\beta + \nabla_\beta T_\alpha$ of the vector field $T = \partial_t$ satisfy

$$|\pi_{\alpha\beta}^{(T)}| \leq \frac{1}{(1 + |t|)^{1+\delta}}$$

for some $\delta > 0$. Show that, for any initial data set $(\psi_0, \psi_1) \in C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3)$ at $\{t = 0\}$, the corresponding solution ψ of the wave equation $\square_g \psi = 0$ satisfies

$$\sup_t \sum_{\alpha=0}^3 \|\partial_\alpha \psi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \leq C_\delta \left(\|\psi_0\|_{H^1} + \|\psi_1\|_{L^2} \right)$$

for some constant $C_\delta > 0$ depending only on δ .

(Hint: Apply the divergence identity for the vector field T .)

11.2 In this exercise, we will establish a number functional inequalities on \mathbb{R}^3 that are frequently used in analysis in order to control L^P norms of a given function by L^q norms of its higher order derivatives. To this end, for any function $f \in C_0^\infty(\mathbb{R}^3)$ and any $k \in \mathbb{N}$, we will set

$$\|\nabla_x^k f\|_{L_x^2(\mathbb{R}^3)}^2 \doteq \sum_{\substack{(a_1, a_2, a_3) \in \mathbb{N}^3: \\ a_1 + a_2 + a_3 = k}} \int_{\mathbb{R}^3} \left| \frac{\partial^{a_1 + a_2 + a_3}}{(\partial x^1)^{a_1} (\partial x^2)^{a_2} (\partial x^3)^{a_3}} f \right|^2 dx.$$

We will also define the Sobolev norm by

$$\|f\|_{H^k(\mathbb{R}^3)}^2 \doteq \sum_{l=0}^k \|\nabla_x^l f\|_{L^2(\mathbb{R}^3)}^2$$

and, for the *homogeneous* Sobolev norm:

$$\|f\|_{\dot{H}^k(\mathbb{R}^3)} \doteq \|\nabla_x^k f\|_{L^2(\mathbb{R}^3)}$$

(note the dot in the notation for \dot{H}^k vs H^k).

(a) Denoting with \hat{f} the Fourier transform of f , i.e.

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot x} f(x) dx,$$

show that there exists a constant $C_k > 0$ depending only on $k \in \mathbb{N}$ such that

$$\frac{1}{C_k} \|\xi|^k \hat{f}\|_{L^2_\xi(\mathbb{R}^3)} \leq \|f\|_{\dot{H}^k(\mathbb{R}^3)} \leq C_k \|\xi|^k \hat{f}\|_{L^2_\xi(\mathbb{R}^3)}.$$

Moreover, show that

$$\|f\|_{L^\infty_x(\mathbb{R}^3)} \leq \|\hat{f}\|_{L^1_\xi(\mathbb{R}^3)}.$$

Hint: For the last inequality, use the formula for the inverse Fourier operator $f(x) = \int_{\mathbb{R}^3} e^{-2\pi i \xi \cdot x} \hat{f}(\xi) d\xi$.

- (b) Show that there exists a constant $C > 0$ such that, for all functions $f \in C_0^\infty(\mathbb{R}^3)$, the following **Sobolev-type** inequality holds:

$$\|f\|_{L^\infty_x(\mathbb{R}^3)}^2 \leq C \cdot \left(\|f\|_{\dot{H}^2_x(\mathbb{R}^3)}^2 + \|f\|_{\dot{H}^1_x(\mathbb{R}^3)}^2 \right)$$

Remark. Note that, contrary to the more standard Sobolev inequality, the right hand side above does not contain a zeroth order term in f . This inequality is useful in the case of wave-type equations, where the naturally conserved energy-type norms usually do not involve undifferentiated terms of the unknown functions.

- (c) Let $\Omega \subset \mathbb{R}^3$ be a *bounded* domain with smooth boundary. We will assume as given the following corollary of the more general Rellich–Kondrachov theorem: For any such domain (in fact, any such domain in \mathbb{R}^n), the identity map $\text{id} : H^1(\Omega) \rightarrow L^2(\Omega)$ is *compact*, i.e. the set of functions $\{f : \|f\|_{H^1(\Omega)} \leq 1\}$ is pre-compact with respect to the L^2 topology. Show that there exists a constant C (depending only on the domain Ω , such that

$$\|f\|_{L^2(\Omega)} \leq C \|\nabla_x f\|_{L^2(\Omega)} \quad \text{for all } f \in C^\infty(\Omega) \text{ such that } \int_\Omega f dx = 0.$$

This is a variant of the **Poincare inequality**.

(*Hint:* Assume, for the sake of contradiction, that there exists a sequence of functions f_n with $\int_\Omega f_n = 0$ such that $\|f_n\|_{L^2(\Omega)} = 1$ and $\|\nabla_x f_n\|_{L^2(\Omega)} \xrightarrow{n \rightarrow \infty} 0$.)

11.3 We will now extend the functional inequalities of the previous exercise to more general 3-dimensional Riemannian manifolds. To this end, let (\mathcal{M}^3, h) be a *compact Riemannian* manifold. We will set for any (k, l) -tensor field T on \mathcal{M} :

$$\|T\|_{L^2(\mathcal{M})}^2 = \int_{\mathcal{M}} \|T\|_h^2 d\text{vol}_h,$$

where

$$\|T\|_h^2 \doteq \langle T, T \rangle_h = h_{i_1 a_1} \dots h_{i_k a_k} h^{j_1 b_1} \dots h^{j_l b_l} T_{j_1 \dots j_l}^{i_1 \dots i_k} T_{b_1 \dots b_l}^{a_1 \dots a_k}$$

and $d\text{vol}_h$ is the natural volume form associated to h so that, in any local coordinate system (x^1, x^2, x^3) , $d\text{vol}_h = \sqrt{\det(h)} dx^1 dx^2 dx^3$.

- (a) Show that there exists a constant $C > 0$ (depending on (\mathcal{M}, h)) such that, for any $f \in C^\infty(\mathcal{M})$, we have

$$\|f\|_{L^\infty(\mathcal{M})}^2 \leq C \cdot \left(\|\nabla^2 f\|_{L^2(\mathcal{M})}^2 + \|\nabla f\|_{L^2(\mathcal{M})}^2 + \|f\|_{L^2(\mathcal{M})}^2 \right).$$

Hint: Fix a finite collection of compact sets $\{\mathcal{K}_n\}_n$ such that each \mathcal{K}_n lies strictly inside a coordinate chart and the open sets $\text{int}(\mathcal{K}_n)$ cover all of \mathcal{M} . If $\chi_n : \mathcal{M} \rightarrow [0, +\infty)$ are cutoff-functions such that $\sum_n \chi_n = 1$ and $\text{supp}(\chi_n) \subset \mathcal{K}_n$, use the previous part of the exercise for the function $\chi_n \cdot f$.

- (b) Using the Poincare inequality, show that there exists a $C > 0$ such that, for any $f \in C^\infty(\mathcal{M})$:

$$\|f - \bar{f}\|_{L^\infty(\mathcal{M})}^2 \leq C \cdot \left(\|\nabla^2 f\|_{L^2(\mathcal{M})}^2 + \|\nabla f\|_{L^2(\mathcal{M})}^2 \right),$$

where

$$\bar{f} \doteq \int_{\mathcal{M}} f \, \text{dvol}_h.$$

11.4 In this exercise, we will use the functional inequalities established earlier to deduce a general boundedness statement for solutions to the wave equation on static spacetimes.

Let (\mathcal{M}^3, h) be a compact Riemannian manifold and let us consider the (static) Lorentzian metric

$$g = -dt^2 + h$$

on $\tilde{\mathcal{M}} = \mathbb{R} \times \mathcal{M}$, where t is the projection on the first factor of $\mathbb{R} \times \mathcal{M}$. Let also T denote the Killing vector field corresponding to translations in the \mathbb{R} factor (so that, in any local coordinate system of the form (t, x^1, x^2, x^3) on $\tilde{\mathcal{M}}$, where (x^1, x^2, x^3) are coordinates on \mathcal{M} , we have that $T = \frac{\partial}{\partial t}$). For any function $\psi \in C^\infty(\tilde{\mathcal{M}})$, we will set

$$\mathcal{E}[\psi](t) \doteq \int_{\{t\} \times \mathcal{M}} \left(|T(\psi)|^2 + \|\bar{\nabla} \psi\|_h^2 \right) \text{dvol}_h,$$

where $\bar{\nabla}$ denotes the connection of h (so that in (t, x^1, x^2, x^3) coordinates as before, $\bar{\nabla} \psi = \partial_i \psi dx^i$).

Let ψ be a smooth solution of the wave equation $\square_g \psi = 0$ on $\tilde{\mathcal{M}}$. In this exercise, we will see how the energy estimates can be used to obtain estimates on the global-in-time behaviour of ψ .

- (a) Show that, for any $t \in \mathbb{R}$,

$$\mathcal{E}[\psi](t) = \mathcal{E}[\psi](0).$$

- (b) Show that $T\psi$ is also a solution of the wave equation. Moreover, show that, for any $t \in \mathbb{R}$,

$$\int_{\{t\} \times \mathcal{M}} (\Delta_h \psi)^2 \, d\text{vol}_h \leq \mathcal{E}[T\psi](t),$$

where Δ_h is the elliptic Laplace-Beltrami operator associated to h :

$$\Delta_h = h^{ab} \bar{\nabla}_a \bar{\nabla}_b \psi$$

(Hint: Use the wave equation for ψ to get an expression for $T^2\psi$.)

- *(c) Show that there exists a constant $C > 0$ depending only on (\mathcal{M}, h) such that

$$\int_{\{t\} \times \mathcal{M}} |\bar{\nabla}^2 \psi|^2 \, d\text{vol}_h \leq C \left(\mathcal{E}[T\psi](t) + \mathcal{E}[\psi](t) \right)$$

Hint: Use integration by parts in the expression

$$\int_{\{t\} \times \mathcal{M}} (\Delta_h \psi)^2 \, d\text{vol}_h = \int_{\{t\} \times \mathcal{M}} h^{ab} h^{ij} \bar{\nabla}_a \bar{\nabla}_b \psi \bar{\nabla}_i \bar{\nabla}_j \psi \, d\text{vol}_h,$$

noting that

$$\int_{\{t\} \times \mathcal{M}} |\bar{\nabla}^2 \psi|^2 \, d\text{vol}_h = \int_{\{t\} \times \mathcal{M}} h^{ab} h^{ij} \bar{\nabla}_a \bar{\nabla}_i \psi \bar{\nabla}_b \bar{\nabla}_j \psi \, d\text{vol}_h.$$

- (d) Show that there exists a constant $C > 0$ depending on (\mathcal{M}, h) such that

$$\sup_t \|\psi - \bar{\psi}\|_{L^\infty(\{t\} \times \mathcal{M})}^2 \leq C \cdot \left(\mathcal{E}[\psi](0) + \mathcal{E}[T\psi](0) \right),$$

where

$$\bar{\psi}(t) \doteq \int_{\{t\} \times \mathcal{M}} \psi \, d\text{vol}_h.$$

- (e) Show that $\bar{\psi}(t)$ satisfies $\frac{d^2}{dt^2} \bar{\psi} = 0$. Deduce that if $\int_{t=0} (T\psi) \, d\text{vol}_h = 0$, then

$$\sup_{\tilde{\mathcal{M}}} |\psi| < +\infty.$$

What if $\int_{t=0} (T\psi) \, d\text{vol}_h \neq 0$?